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SOLVING THE PUNCH PROBLEMS BY ANALOGY WITH THE INTERFACE CRACK PROBLEMS

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Abstract—By cutting through the contact/crack surface, the contact/crack problems can be treated as a half-plane problem with the displacements/tractions prescribed along a specified region. Through the conversion to half-plane problems, the analogy between the contact problems and crack problems has been noticed before. However, to the authors' knowledge, no detailed analogous relations have been provided. Moreover, no contact problems have been solved directly by just using the corresponding solutions of the crack problems, or vice versa. In this paper, special consideration will be focused upon the connections between the punch problems and the interface crack problems with one of the materials to be rigid. Similar to the analogy between line forces and line dislocations, cracks and rigid line inclusions, or holes and rigid inclusions, we may now solve the punch problems by analogy with the interface crack problems. In addition, very simple explicit solutions for the contact pressure and the surface deformation are derived in this paper. Moreover, a general procedure to get the real form solution is also described. Finally, three representative punch problems are solved completely. They are the indentation by a flat-ended punch, a flat-ended punch tilted by a couple, and the indentation by a parabolic punch. The explicit full field solutions, and the real form solutions for the contact pressure and surface deformation of these three problems are all provided. Based upon these closed-form solutions, several numerical examples were done and their related stresses contours, surface deformations and contact pressures were also plotted to help us see more clearly the physical behaviors of the punch problems.

1. INTRODUCTION

Solving problems by analogy techniques is not unusual. In the solution of torsional problems the membrane analogy has been proved very valuable (Timoshenko and Goodier, 1970). Their governing equations and boundary conditions are identical in the form of mathematical expressions, but their symbols have different physical meanings. Therefore, by replacing the symbols, they can communicate with each other. Other kinds of analogic problems known in the linear elastic are force and dislocation, crack and rigid line inclusion, hole and rigid inclusion, etc. Their governing equations are the same, but their boundary conditions are different in the sense that one is traction-prescribed the other is displacement-prescribed. Due to the fact that the traction-prescribed and displacement-prescribed boundary conditions have similar mathematical expressions, they can also benefit from analogy technique.

As to the contact problems and the crack problems, similar analogy has been noticed before, e.g. (Willis, 1968; Brock, 1978). However, to the authors' knowledge, these two kinds of problems were solved independently and no detailed analogous relations have been provided. In this paper, the analogy will be discussed for the most general anisotropic linear elastic materials and the contact problems will be solved directly by just using the corresponding solutions of crack problems. The contact problems considered here will be restricted to the two-dimensional cases in which the indentor is a rigid punch. By this restriction, the materials above the interface crack will also be rigid. Through this study, it is hoped that the analogy technique can be extended to the general contact and interface crack problems. Moreover, it is also possible to explore the analogy between the three-dimensional contact and crack problems since both of these two problems were usually

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formulated as problems of Boussinesq type (Sneddon, 1946; Sneddon and Lowengrub, 1969; Willis, 1996, 1967, 1968, 1970; Gladwell, 1980).

2. TWO-DIMENSIONAL ANISOTROPIC ELASTICITY

The basic equations for linear anisotropic elasticity are the strain–displacement equations, the stress–strain laws and the equations of equilibrium, which can be expressed in a fixed rectangular coordinate system x_i, $i = 1, 2, 3$ as (the symbols x_1 and x_2 will be replaced by x and y for the convenience of presentation)

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \quad \sigma_{ij} = C_{ijks}\varepsilon_{ks}, \quad \sigma_{ij,j} = C_{ijks}u_{k,sj} = 0, \tag{2.1}$$

where u_i, σ_{ij} and ε_{ij}, are, respectively, the displacement, stress and strain; the repeated indices imply summation; a comma stands for differentiation and C_{ijks} are the elastic constants which are assumed to be fully symmetric and positive definite.

For two-dimensional problems in which x_3 does not appear in the basic equations or the boundary conditions, the general solution to eqns (2.1) may be expressed in terms of three holomorphic functions of complex variables (Stroh, 1958; Lekhnitskii, 1963). This enables us to apply many of the powerful results of complex function theory to the two-dimensional elasticity. For the later use of derivation, we now list a compact matrix form solution (Stroh, 1958; Ting, 1986) which satisfies all the basic equations given in (2.1), i.e.,

$$\begin{aligned} \underset{\sim}{u} &= 2\,Re\,\{\underset{\sim}{A}\underset{\sim}{f}(z)\} = \underset{\sim}{A}\underset{\sim}{f}(z)+\overline{\underset{\sim}{A}}\overline{\underset{\sim}{f}(z)}, \\ \underset{\sim}{\phi} &= 2\,Re\,\{\underset{\sim}{B}\underset{\sim}{f}(z)\} = \underset{\sim}{B}\underset{\sim}{f}(z)+\overline{\underset{\sim}{B}}\overline{\underset{\sim}{f}(z)}, \end{aligned} \tag{2.2a}$$

where

$$\begin{aligned} &\underset{\sim}{A} = [\underset{\sim}{a}_1 \quad \underset{\sim}{a}_2 \quad \underset{\sim}{a}_3], \quad \underset{\sim}{B} = [\underset{\sim}{b}_1 \quad \underset{\sim}{b}_2 \quad \underset{\sim}{b}_3], \\ &\underset{\sim}{f}(z) = [f_1(z_1) \quad f_2(z_2) \quad f_3(z_3)]^T, \quad z_\alpha = x+p_\alpha y, \quad \alpha = 1,2,3. \end{aligned} \tag{2.2b}$$

In the above equations, $\underset{\sim}{u} = (u_1, u_2, u_3)$ is the vector form of displacement; $\underset{\sim}{\phi} = (\phi_1, \phi_2, \phi_3)$ stands for the stress function vector which is related to the stresses by

$$\sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1}; \tag{2.2c}$$

p_α, $\alpha = 1, 2, 3$, are the material eigenvalues whose imaginary parts have been arranged to be positive; $(\underset{\sim}{a}_\alpha, \underset{\sim}{b}_\alpha)$, $\alpha = 1, 2, 3$, are their associated eigenvectors; $f_\alpha(z_\alpha)$, $\alpha = 1, 2, 3$, are three holomorphic complex functions to be determined by satisfying the boundary conditions of the problems considered. The superscript T denotes the transpose and the overbar represents the conjugate of a complex number.

Note that (Suo, 1990; Hwu, 1993a) during the derivation through the method of analytical continuation (Muskhelishvili, 1954), the argument of each component function of $f(z)$ is written as $z = x+py$ without referring to its associated eigenvalues p_α. Once the solution of $\underset{\sim}{f}(z)$ is obtained for a given boundary value problem, a replacement of z_1, z_2 or z_3 should be made for each component function to calculate field quantities from (2.2). In other words, the function vector $\underset{\sim}{f}(z)$ obtained through the method of analytical continuation has the form of

$$\underset{\sim}{f}(z) = [f_1(z), f_2(z), f_3(z)]^T, \quad z = x+py, \tag{2.3}$$

which is not consistent with the solution form shown in (2.2b) and is valid only along the boundary $y = 0$. To get the explicit full domain solution, a mathematical operation based upon the above statement is needed. A translating technique presented by Hwu (1993a) is then introduced as below.

If an implicit solution is written as

$$\underset{\sim}{f}(z) = \underset{\sim}{C}\langle\langle g_\alpha(z)\rangle\rangle \underset{\sim}{q}, \tag{2.4a}$$

with the understanding the subscript of z is dropped before matrix product and a replacement of z_1, z_2 or z_3 should be made for each component function of $\underset{\sim}{f}(z)$ after the multiplication of matrices, the explicit solution can be expressed as

$$\underset{\sim}{f}(z) = \sum_{k=1}^{3} \langle\langle g_k(z_\alpha)\rangle\rangle \underset{\sim}{C}\underset{\sim}{I}_k \underset{\sim}{q}, \tag{2.4b}$$

where

$$\underset{\sim}{I}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underset{\sim}{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underset{\sim}{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.4c}$$

In the above, the angular brackets $\langle\langle\,\rangle\rangle$ stands for the diagonal matrix in which each component is varied according to the Greek index α, i.e. $\langle\langle g_k(z_\alpha)\rangle\rangle = \mathrm{diag}\,[g_k(z_1), g_k(z_2), g_k(z_3)]$.

3. KNOWN ANALOGIC PROBLEMS

One of the special features of the Stroh's formalism is that the solution form (2.2) is neat and elegant. Due to its elegancy, many important characteristics can be found at the first glance of the solution form. For example, the displacements and stress functions shown in (2.2a) are distinguished only by the material eigenvector matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$. Thus, the relevant boundary conditions of the displacement prescribed problems differ from those of the traction prescribed problems only in the appearance of the symbols $\underset{\sim}{A}$ and $\underset{\sim}{B}$. Since the mathematical formulations for the displacement prescribed problems and the traction prescribed problems are identical, their solution should also be identical with $\underset{\sim}{A}$ and $\underset{\sim}{B}$ interchanged. In the following, we list some known analogic problems to help the readers see more clearly what we try to express.

Line force and line dislocation

Consider an infinite homogeneous anisotropic elastic space. The boundary conditions for a line force $\hat{\underset{\sim}{f}}$/dislocation $\hat{\underset{\sim}{b}}$ applied at $(\hat{x}_1, \hat{x}_2)$ along the x_3-axis may be expressed as

$$\text{line force:} \quad \int_C \mathrm{d}\underset{\sim}{\phi} = \hat{\underset{\sim}{f}} \quad \text{for any close curve } C \text{ enclosing } \hat{\underset{\sim}{x}},$$

$$\text{line dislocation:} \quad \int_C \mathrm{d}\underset{\sim}{u} = \hat{\underset{\sim}{b}} \quad \text{for any close curve } C \text{ enclosing } \hat{\underset{\sim}{x}}, \tag{3.1}$$

and $\sigma_{ij} \to 0$ at infinity. The solution to these two problems are (Eshelby *et al.*, 1953)

$$\text{line force:} \quad \mathbf{f}(z) = \frac{1}{2\pi i} \langle\langle \ln(z_\alpha - \hat{z}_\alpha) \rangle\rangle \mathbf{A}^T \hat{\mathbf{t}},$$

$$\text{line dislocation:} \quad \mathbf{f}(z) = \frac{1}{2\pi i} \langle\langle \ln(z_\alpha - \hat{z}_\alpha) \rangle\rangle \mathbf{B}^T \hat{\mathbf{b}}. \tag{3.2}$$

Crack and rigid line inclusion

Consider a crack/rigid line inclusion of length of $2a$ centrally located at $y = 0$ and $|x| < a$ in a homogeneous anisotropic elastic medium. If the stresses vanish at infinity and the crack surface is subjected to a uniform traction $\hat{\mathbf{t}}$ and the rigid line inclusion is prescribed by a uniform strain $\hat{\boldsymbol{\varepsilon}}$, the boundary conditions may be expressed as

$$\text{crack:} \quad \boldsymbol{\phi}_{,1} = \hat{\mathbf{t}}, \quad \text{for } y = \pm 0 \quad \text{and} \quad -a < x < a$$

$$\text{rigid line inclusion:} \quad \mathbf{u}_{,1} = \hat{\boldsymbol{\varepsilon}}, \quad \text{for } y = \pm 0 \quad \text{and} \quad -a < x < a \tag{3.3}$$

and $\sigma_{ij} \to 0$ at infinity. The solution to these two problems are (Ting, 1996)

$$\text{crack:} \quad \mathbf{f}(z) = -\langle\langle z_\alpha - \sqrt{z_\alpha^2 - a^2} \rangle\rangle \mathbf{B}^{-1} \hat{\mathbf{t}},$$

$$\text{rigid line inclusion:} \quad \mathbf{f}(z) = -\langle\langle z_\alpha - \sqrt{z_\alpha^2 - a^2} \rangle\rangle \mathbf{A}^{-1} \hat{\boldsymbol{\varepsilon}}. \tag{3.4}$$

Hole and rigid inclusion

Consider an infinite anisotropic plate containing an elliptic hole/rigid inclusion under a concentrated force $\hat{\mathbf{t}}$ applied at point $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$. If the hole is assumed to be traction free and the rigid inclusion is assumed to be perfectly bonded to the matrix, the boundary conditions can be written as

$$\text{hole:} \quad \boldsymbol{\phi}_{,n} = \mathbf{0} \quad \text{along the hole boundary,}$$

$$\int_C \mathrm{d}\boldsymbol{\phi} = \hat{\mathbf{t}} \quad \text{for any close curve } C \text{ enclosing } \hat{\mathbf{x}},$$

$$\text{rigid inclusion:} \quad \mathbf{u}_{,n} = \mathbf{0} \quad \text{along the inclusion boundary,}$$

$$\int_C \mathrm{d}\boldsymbol{\phi} = \hat{\mathbf{t}} \quad \text{for any close curve } C \text{ enclosing } \hat{\mathbf{x}}, \tag{3.5}$$

and $\sigma_{ij} \to 0$ at infinity. The subscript $(,n)$ means differentiation with respect to the tangent direction of the boundary. The solution to these two problems are (Hwu and Yen, 1993)

$$\text{hole:} \quad \mathbf{f}(\zeta) = \mathbf{f}_0(\zeta) + \frac{1}{2\pi i} \sum_{k=1}^{3} \langle\langle \log(\zeta_\alpha^{-1} - \bar{\hat{\zeta}}_k) \rangle\rangle \mathbf{B}^{-1} \bar{\mathbf{B}} \mathbf{I}_k \bar{\mathbf{A}}^T \hat{\mathbf{t}},$$

$$\text{rigid inclusion:} \quad \mathbf{f}(\zeta) = \mathbf{f}_0(\zeta) + \frac{1}{2\pi i} \sum_{k=1}^{3} \langle\langle \log(\zeta_\alpha^{-1} - \bar{\hat{\zeta}}_k) \rangle\rangle \mathbf{A}^{-1} \bar{\mathbf{A}} \mathbf{I}_k \bar{\mathbf{A}}^T \hat{\mathbf{t}}. \tag{3.6a}$$

where

$$\mathbf{f}_0(\zeta) = \langle\langle \log(\zeta_\alpha - \hat{\zeta}_\alpha) \rangle\rangle \mathbf{A}^T \hat{\mathbf{t}}, \tag{3.6b}$$

$$\zeta_\alpha = \frac{z_\alpha + \sqrt{z_\alpha^2 - (a^2 + p_\alpha^2 b^2)}}{a - i p_\alpha b}, \quad \hat{\zeta}_k = \frac{\hat{z}_k + \sqrt{\hat{z}_k^2 - (a^2 + p_k^2 b^2)}}{a - i p_k b}, \tag{3.6c}$$

and a, b are the half length of the major and minor axes of the ellipse.

4. SOLVING THE PUNCH PROBLEMS BY ANALOGY TECHNIQUE

At the first glance, it is not easy to see any connection between the punch problems and the collinear interface crack problems. Fortunately, we solved these two seemingly different problems individually by using the Stroh's formalism and the method of analytical continuation (Hwu, 1993a; Fan and Hwu, 1996). With the obtained solutions, we saw some similar behaviors like the stress oscillatory characteristics near the interface crack tips or the ends of the flat-ended punches. This similarity stimulates us to find the connection between these two problems. By carefully reviewing these two different boundary conditions, we find a simple way to solve the punch problems by analogy with the collinear interface crack problems.

Consider the case that a set of rigid punches of given profiles are brought into contact with the surface of the half-plane and are allowed to indent the surface in such a way that the punches completely adhere to the half-plane on initial contact and during the subsequent indentation no slip occurs and the contact region does not change. Let us suppose the contact region L is the union of a finite set of line segments $L_k = (a_k, b_k)$, $k = 1, 2, \ldots, n$, where the ends of the segments are encountered in the order $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ when moving in the positive x-direction. For this case the displacements of the surface of the half-plane are known at each point of the contact region, then the boundary conditions are

$$\underset{\sim}{u}(x) = (u_k(x), v_k(x)+c_k, 0)^T = \hat{\underset{\sim}{u}}(x), \quad x \in L,$$
$$\underset{\sim}{t}(x) = (\sigma_{xy}, \sigma_{yy}, \sigma_{zy})^T = \underset{\sim}{\phi}'(x) = \underset{\sim}{0}, \quad x \notin L, \tag{4.1}$$

where $u_k(x)$ and $v_k(x)$ are related to the profile of the kth punch and c_k is the relative depth of indentation.

In order to solve the punch problems by using analogy technique, we now examine the boundary conditions of the collinear interface crack problems. The prescribed-traction condition on the crack portion, as well as the continuity of displacements and tractions across the bonded portion of the interface may be written as (Hwu, 1993a)

$$\underset{\sim}{\phi}_1'(x) = \underset{\sim}{\phi}_2'(x) = \hat{\underset{\sim}{t}}, \quad x \in L,$$
$$\underset{\sim}{u}_1(x) = \underset{\sim}{u}_2(x), \quad \underset{\sim}{\phi}_1(x) = \underset{\sim}{\phi}_2(x), \quad x \notin L, \tag{4.2}$$

where prime (′) denotes differentiation with respect to its argument. The symbols marked with the subscripts 1 and 2 represents, respectively, the quantities pertaining to the materials located upper and lower the interface. The solution satisfying the boundary conditions set in (4.2) has been found by using the Stroh's formalism and the method of analytical continuation (Hwu, 1993a). The results are

$$\underset{\sim}{f}_1(z) = \underset{\sim}{B}_1^{-1}\underset{\sim}{\psi}(z), \quad \underset{\sim}{f}_2(z) = \underset{\sim}{B}_2^{-1}\bar{\underset{\sim}{M}}^{*-1}\underset{\sim}{M}^*\underset{\sim}{\psi}(z), \tag{4.3a}$$

where

$$\underset{\sim}{\psi}'(z) = \frac{1}{2\pi i}\underset{\sim}{X}_0^*(z)\int_L \frac{1}{s-z}[\underset{\sim}{X}_0^{*+}(s)]^{-1}\hat{\underset{\sim}{t}}(s)\,\mathrm{d}s + \underset{\sim}{X}_0^*(z)\underset{\sim}{p}_n(z). \tag{4.3b}$$

In the above, $p_n(z)$ is an arbitrary polynomial vector with the degree not higher than the number of cracks n, which may be determined by the infinity conditions and the single-valuedness requirement of displacements. $\underset{\sim}{M}^*$ is the bimaterial matrix defined as

$$\underset{\sim}{M}^* = \underset{\sim}{D} - i\underset{\sim}{W}, \quad \underset{\sim}{D} = \underset{\sim}{L}_1^{-1} + \underset{\sim}{L}_2^{-1}, \quad \underset{\sim}{W} = \underset{\sim}{S}_1\underset{\sim}{L}_1^{-1} - \underset{\sim}{S}_2\underset{\sim}{L}_2^{-1}, \tag{4.4}$$

where $\underset{\sim}{S}_i$ and $\underset{\sim}{L}_i$ are 3×3 real matrices composed of the elasticity constants. They are defined by $\underset{\sim}{S}_i = i(2\underset{\sim}{A}_i\underset{\sim}{B}_i^T - \underset{\sim}{I})$, $\underset{\sim}{L}_i = -2i\underset{\sim}{B}_i\underset{\sim}{B}_i^T$, $i = 1, 2$ (Ting, 1988). $\underset{\sim}{X}_0^*(z)$ is the basic Plemelj function matrix satisfying

$$\underset{\sim}{X}{}^{*+}_0(x) = \underset{\sim}{X}{}^{*-}_0(x), \quad x \notin L,$$
$$\underset{\sim}{X}{}^{*+}_0(x) + \bar{\underset{\sim}{M}}{}^{*-1}\underset{\sim}{M}^*\underset{\sim}{X}{}^{*-}_0(x) = \underset{\sim}{0}, \quad x \in L. \tag{4.5}$$

Since the punches are assumed to be rigid, only one material, i.e. the half-plane to be indented, is considered in the boundary conditions (4.1). For the collinear interface crack problems, two different materials are considered in the boundary conditions (4.2). Without further studying, it looks no connections between (4.1) and (4.2). However, if we consider the material above the interface to be rigid, the boundary conditions (4.2) will then be reduced to

$$\underset{\sim}{\phi}'(x) = \hat{\underset{\sim}{t}}(x), \quad x \in L,$$
$$\underset{\sim}{u}(x) = \underset{\sim}{0}, \quad x \notin L. \tag{4.6}$$

Note that in (4.6) and the following derivation the subscript 2 is dropped for the convenience of presentation, and the subscript 1 will not enter into the boundary conditions since material 1 is assumed to be rigid.

Comparison between (4.1) and (4.6), we see that (4.6) is just a counterpart of (4.1) since the traction prescribed condition $\underset{\sim}{\phi}' = \hat{\underset{\sim}{t}}$ of (4.6) corresponds to the displacement prescribed condition $\underset{\sim}{u}' = \hat{\underset{\sim}{u}}'$ of (4.1), and the displacement prescribed condition $\underset{\sim}{u}' = \underset{\sim}{0}$ of (4.6) corresponds to the traction prescribed condition $\underset{\sim}{\phi}' = \underset{\sim}{0}$ of (4.1). Therefore, we may solve the punch problems by using the solutions of the collinear interface crack problems, eqns (4.3)–(4.5), with material 1 taken to be rigid, and interchanging the material eigenvector matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$. With this concept in mind, we now show the detailed mathematical derivation and compare the results with the solution found in (Fan and Hwu, 1996).

It is known that all the components of the real matrix $\underset{\sim}{L}$ is proportional to the Young's modulus (Hwu, 1993b). If material 1 is rigid, $\underset{\sim}{L}_1^{-1}$ of (4.4) will vanish, and $\underset{\sim}{D} = \underset{\sim}{L}^{-1}$, $\underset{\sim}{W} = -\underset{\sim}{S}\underset{\sim}{L}^{-1}$. The bimaterial matrix $\underset{\sim}{M}^*$ defined in (4.4) then becomes

$$\underset{\sim}{M}^* = (\underset{\sim}{I} + i\underset{\sim}{S})\underset{\sim}{L}^{-1} = \bar{\underset{\sim}{M}}{}^{-1}, \tag{4.7}$$

where $\underset{\sim}{M}$ is the impedance matrix defined as (Ting, 1988),

$$\underset{\sim}{M} = -i\underset{\sim}{B}\underset{\sim}{A}^{-1} = \underset{\sim}{L}(\underset{\sim}{I} - i\underset{\sim}{S})^{-1}. \tag{4.8}$$

Substituting (4.7) into (4.3) and (4.5), we have

$$\underset{\sim}{f}(z) = \underset{\sim}{B}^{-1}\underset{\sim}{M}\bar{\underset{\sim}{M}}{}^{-1}\underset{\sim}{\psi}(z),$$
$$\underset{\sim}{\psi}'(z) = \frac{1}{2\pi i}\underset{\sim}{X}_0^*(z)\int_L \frac{1}{s-z}[\underset{\sim}{X}{}^{*+}_0(s)]^{-1}\hat{\underset{\sim}{t}}(s)\,\mathrm{d}s + \underset{\sim}{X}_0^*(z)\underset{\sim}{p}_n(z), \tag{4.9a}$$

where

$$\underset{\sim}{X}{}^{*+}_0(x) = \underset{\sim}{X}{}^{*-}_0(x), \quad x \notin L,$$
$$\underset{\sim}{X}{}^{*+}_0(x) + \underset{\sim}{M}\bar{\underset{\sim}{M}}{}^{-1}\underset{\sim}{X}{}^{*-}_0(x) = \underset{\sim}{0}, \quad x \in L. \tag{4.9b}$$

(4.9) is the solution for the collinear interface crack problems with material 1 taken to be rigid. To get the solution for the punch problems, we now interchange $\underset{\sim}{A}$ and $\underset{\sim}{B}$ (which also leads to the interchange of $\underset{\sim}{M}$ and $-\underset{\sim}{M}^{-1}$) and replace $\hat{\underset{\sim}{t}}$ by $\hat{\underset{\sim}{u}}'$. The results are

$$f(z) = A^{-1} M^{-1} \bar{M} \psi(z),$$

$$\psi'(z) = \frac{1}{2\pi i} X_0^*(z) \int_L \frac{1}{s-z} [X^{*+}_0(s)]^{-1} \hat{u}'(s)\, ds + X_0^*(z) p_n(z), \tag{4.10a}$$

where

$$X^{*+}_0(x) = X^{*-}_0(x), \quad x \notin L,$$

$$X^{*+}_0(x) + M^{-1} \bar{M} X^{*-}_0(x) = 0, \quad x \in L. \tag{4.10b}$$

In order to compare with the solutions obtained in (Fan and Hwu, 1996), we make the following rearrangement,

$$f(z) = B^{-1}\theta(z), \quad \theta(z) = i\bar{M}\psi(z), \quad X_0(z) = \bar{M}X_0^*(z). \tag{4.11}$$

The solution shown in (4.10) may now be rewritten as

$$f(z) = B^{-1}\theta(z),$$

$$\theta'(z) = \frac{1}{2\pi} X_0(z) \int_L \frac{1}{s-z} [X_0^+(s)]^{-1} \bar{M} \hat{u}'(s)\, ds + X_0(z) p_n(z), \tag{4.12a}$$

where

$$X_0^+(x) = X_0^-(x), \quad x \notin L,$$

$$X_0^+(x) + \bar{M} M^{-1} X_0^-(x) = 0, \quad x \in L, \tag{4.12b}$$

which is exactly the same as that shown in (Fan and Hwu, 1996). The Plemelj function matrix $X_0(z)$ satisfying (4.12b) can also be obtained by analogy with the collinear interface crack problems (Hwu, 1993a) with M^* replaced by M^{-1}. This replacement is due to the comparison of (4.12b) and (4.5). One should be very careful not to take (4.7) for the replacement since its related solution (4.9b) is not for punch problems. The solution to (4.12b) is

$$X_0(z) = \Lambda\Gamma(z), \tag{4.13a}$$

where

$$\Lambda \doteq [\lambda_1, \lambda_2, \lambda_3], \quad \Gamma(z) = \left\langle\left\langle \prod_{j=1}^{n} \frac{1}{\sqrt{(z-a_j)(z-b_j)}} \left(\frac{z-b_j}{z-a_j}\right)^{i\varepsilon_\alpha} \right\rangle\right\rangle. \tag{4.13b}$$

ε_α and λ_α, $\alpha = 1, 2, 3$ of (4.13b) are the eigenvalues and eigenvectors of $(M^{-1} - e^{-2\pi\varepsilon_\alpha}\bar{M}^{-1})\lambda_\alpha = 0$. The explicit solutions for the eigenvalues ε_α are $\varepsilon_1 = -\varepsilon_2 = \varepsilon$, $\varepsilon_3 = 0$, and

$$\varepsilon = \frac{1}{2\pi} \ln \frac{1+\beta}{1-\beta}, \quad \beta = \left[-\frac{1}{2}\mathrm{tr}(S^2)\right]^{1/2}, \tag{4.14}$$

where tr stands for the trace of matrix. Moreover, for normalizing the eigenvector matrix Λ, the normalization proposed by Hwu (1993a) may be replaced by $\frac{1}{2}\bar{\Lambda}^T(M^{-1} + \bar{M}^{-1})\Lambda = I$.

Note that $p_n(z)$ in (4.12a) differs from that in (4.10a) by a constant matrix $\bar{M}^{-1}$. However, since $p_n(z)$ is an undetermined polynomial vector, it makes no difference to change its undetermined coefficients. For the convenience of presentation, we keep the same symbol $p_n(z)$ in (4.12a), and expand $p_n(z)$ as $p_n(z) = d_0 + d_1 z + \cdots + d_{n-1} z^{n-1}$. The unknown coefficients of $p_n(z)$ may then be determined by the infinity condition which leads to (Fan and Hwu, 1996)

$$d_{n-1} = \frac{1}{2\pi i}\Lambda^{-1} q, \tag{4.15}$$

and the force conditions

$$q_k = -\int_{L_k} [\theta'(x^+) - \theta'(x^-)]\,\mathrm{d}x, \quad \text{for } k = 1, 2, \ldots, n, \tag{4.16}$$

where q_k is the known resultant force vector applied on the kth punch, and $q = \Sigma_{k=1}^{n} q_k$.

5. CONTACT PRESSURE AND SURFACE DEFORMATION

In engineering application, it is always interesting to know the contact pressure under the punches and the surface deformation outside the punches. By using some identities (Ting, 1988), very simple explicit solutions for the contact pressure and the surface deformation, which were not shown in the literature, will be derived in this section.

The tractions and displacements along the surface of the half-plane are related to $\theta(z)$ of (4.12a) by (Fan and Hwu, 1996)

$$\begin{aligned} i\bar{M}u'(x) &= \theta'(x^+) + \bar{M}M^{-1}\theta'(x^-), \\ t(x) &= \theta'(x^-) - \theta'(x^+). \end{aligned} \tag{5.1}$$

Contact pressure

To find a simplified expression for the contact pressure, we substitute the displacement prescribed condition $(4.1)_1$ into $(5.1)_1$, which leads to

$$\theta'(x^+) = \bar{M}M^{-1}\theta'(x^-) - i\bar{M}\hat{u}'(x), \quad x \in L. \tag{5.2}$$

Substituting (5.2) into $(5.1)_2$, and using the identity shown in $(4.8)_2$, we obtain

$$t(x) = 2\bar{M}L^{-1}\theta'(x^-) - i\bar{M}\hat{u}'(x), \quad x \in L. \tag{5.3}$$

Further reduction may also be done by substituting the definitions of $M(= -iBA^{-1})$ and $L(= -2iBB^T)$ into (5.3), and using the identity $A^T\bar{B} + B^T\bar{A} = 0$ (Ting, 1988). The result is

$$t(x) = A^{-T} f'(x^-) - i\bar{M}\hat{u}'(x), \quad x \in L. \tag{5.4}$$

Surface deformation

For the surface deformations outside the punches, we substitute the traction prescribed condition $(4.1)_2$ into $(5.1)_2$, which leads to

$$\theta'(x^+) = \theta'(x^-), \quad x \notin L. \tag{5.5}$$

Substituting (5.5) into $(5.1)_1$, and using the identity shown in $(4.8)_1$, we obtain

$$\underset{\sim}{u}'(x) = -2i\underset{\sim}{L}^{-1}\underset{\sim}{\theta}'(x^-), \quad x \notin L. \tag{5.6}$$

Similar to (5.4), we may further reduce (5.6) to

$$\underset{\sim}{u}'(x) = \underset{\sim}{B}^{-T}\underset{\sim}{f}'(x^-), \quad x \notin L. \tag{5.7}$$

The simplified expressions shown in (5.3), (5.4) and (5.6), (5.7) are in complex form. In engineering applications, it is always interesting to know the real form solution since both of the stresses and displacements are real quantities. In most cases, the real form solutions can be found by employing the following identities which are obtained also by analogy with the identities found in the interface crack problems (Hwu, 1993b) with $\underset{\sim}{M}^*$ replaced by $\underset{\sim}{M}^{-1}$.

$$\begin{aligned}
\bar{\underset{\sim}{\Lambda}}^T\underset{\sim}{L}^{-1}\underset{\sim}{\Lambda} &= \underset{\sim}{I}, \\
\bar{\underset{\sim}{\Lambda}}^T\underset{\sim}{M}^{-1}\underset{\sim}{\Lambda} &= \langle\langle 1-\tanh(\pi\varepsilon_\alpha)\rangle\rangle, \\
\bar{\underset{\sim}{\Lambda}}^T\bar{\underset{\sim}{M}}^{-1}\underset{\sim}{\Lambda} &= \langle\langle 1+\tanh(\pi\varepsilon_\alpha)\rangle\rangle, \\
\bar{\underset{\sim}{M}}^{-1}\underset{\sim}{L}^{-1}\underset{\sim}{\Lambda} &= \underset{\sim}{\Lambda}\langle\langle e^{-\pi\varepsilon_\alpha}\cosh(\pi\varepsilon_\alpha)\rangle\rangle, \\
\underset{\sim}{\Lambda}\langle\langle c_\alpha\rangle\rangle\underset{\sim}{\Lambda}^{-1} &= \underset{\sim}{I}+\frac{1-c_R}{\beta^2}\underset{\sim}{S}^{T^2}+\frac{c_I}{\beta}\underset{\sim}{S}^T,
\end{aligned} \tag{5.8}$$

where $c_1 = c$, $c_2 = \bar{c}$, $c_3 = 1$ and c_R, c_I are real and imaginary parts of c which is an arbitrary complex number.

6. THREE REPRESENTATIVE PUNCH PROBLEMS

In this section, three representative punch problems will be solved completely. They are the indentation by a flat-ended punch, a flat-ended punch tilted by a couple, and the indentation by a parabolic punch. The first two problems have been studied in our previous paper (Fan and Hwu, 1996). In that paper, we provided the solutions for the contact pressure and its reduction to the orthotropic and isotropic half-planes. To have a complete picture for the punch problems, in this section we would like to supplement the explicit full field solution for the stresses and displacements and the real form solutions for the contact pressure and surface deformation. The third problem is newly solved, therefore, the detailed derivation for the explicit full field solution, contact pressure under the punch and surface deformation outside the punch will all be given. Based upon these closed-form solutions, several numerical examples were done and their related stresses contours, surface deformations and contact pressures were also plotted to help us see more clearly the physical behaviours of the punch problems.

(i) *Indentation by a flat-ended punch*

Consider the case of indentation by a single punch ($n = 1$) with a flat-ended profile ($\hat{\underset{\sim}{u}}' = \underset{\sim}{0}$) which makes contact with the half-plane over the region $|x| \leqslant a$, and the force $\hat{\underset{\sim}{p}}$ applied on the punch is given. From $(4.12a)_2$, (4.13) and (4.15) we find

$$\underset{\sim}{\theta}'(z) = \frac{1}{2\pi i}\underset{\sim}{\Lambda}\underset{\sim}{\Gamma}(z)\underset{\sim}{\Lambda}^{-1}\hat{\underset{\sim}{p}}. \tag{6.1a}$$

where

$$\underset{\sim}{\Gamma}(z) = \left\langle\left\langle \frac{1}{\sqrt{z^2-a^2}}\left(\frac{z+a}{z-a}\right)^{-i\varepsilon_\alpha} \right\rangle\right\rangle. \tag{6.1b}$$

By (6.1), (4.12a)$_1$ and (2.4), the explicit full field solution can be expressed as

$$\underset{\sim}{f}'(z) = \frac{1}{2\pi i}\sum_{k=1}^{3} \langle\langle \Gamma_k(z_\alpha)\rangle\rangle \underset{\sim}{B}^{-1}\underset{\sim}{\Lambda}\underset{\sim}{I}_k\underset{\sim}{\Lambda}^{-1}\underset{\sim}{\hat{p}}. \tag{6.2}$$

To find the contact pressure and the surface deformations, we need to calculate $\underset{\sim}{\Gamma}(x^-)$ for $|x| \leqslant a$ and $|x| > a$. This can easily be evaluated by using a bipolar coordinate system with two origins located at the ends of the punch and introducing a cut along the punch region. The results are

$$\underset{\sim}{\Gamma}(x^-) = \left\langle\left\langle \frac{ie^{\pi\varepsilon_\alpha}}{\sqrt{a^2-x^2}} e^{-i\varepsilon_\alpha \ln\frac{a+x}{a-x}} \right\rangle\right\rangle, \quad \text{for } |x| \leqslant a,$$

$$\underset{\sim}{\Gamma}(x^-) = \left\langle\left\langle \pm\frac{1}{\sqrt{x^2-a^2}} e^{-i\varepsilon_\alpha \ln\left|\frac{x+a}{x-a}\right|} \right\rangle\right\rangle, \quad \text{for } x > a \quad \text{and} \quad x < -a. \tag{6.3}$$

Substituting (6.1) and (6.3) into (5.3) and (5.6), we may get the solutions for the contact pressure under the punch and the surface deformation outside the punch. Since the stresses and deformations are real quantities, it is of interest to obtain the real form solutions in order to have a better understanding of the physical behavior of the punch problems. By using the identities given in (5.8), the real form solution for the contact pressure under the punch and surface deformation gradient outside the punch can be found to be

$$\underset{\sim}{t}(x) = \frac{1}{\pi\sqrt{a^2-x^2}}\left[\underset{\sim}{I} + \frac{1-c_R}{\beta^2}\underset{\sim}{S}^{T^2} + \frac{c_I}{\beta}\underset{\sim}{S}^T\right]\underset{\sim}{\hat{p}}, \quad |x| \leqslant a,$$

$$\underset{\sim}{u}'(x) = \mp\frac{1}{\pi\sqrt{x^2-a^2}}\underset{\sim}{L}^{-1}\left[\underset{\sim}{I} + \frac{1-c_R^*}{\beta^2}\underset{\sim}{S}^{T^2} + \frac{c_I^*}{\beta}\underset{\sim}{S}^T\right]\underset{\sim}{\hat{p}}, \quad x > a \quad \text{and} \quad x < -a, \tag{6.4a}$$

where

$$c_R + ic_I = \cosh(\pi\varepsilon)e^{-i\varepsilon\ln\frac{a+x}{a-x}}$$

$$c_R^* + ic_I^* = e^{-i\varepsilon\ln\left|\frac{x+a}{x-a}\right|}. \tag{6.4b}$$

The above solutions (6.2) and (6.4) are valid for general anisotropic materials. In principle the explicit full field solution shown in (6.2) are valid only for the nondegenerate materials, that is the material eigenvalues p_α, $\alpha = 1, 2, 3$, are distinct or three independent material eigenvectors $\underset{\sim}{a}_\alpha, \underset{\sim}{b}_\alpha, \alpha = 1, 2, 3$, can be found when p_α are repeated. By using a correspondence relation between anisotropic and isotropic elasticity (Hwu, 1996), an analytical solution for isotropic materials deduced from (6.2) is obtained and is proved to be identical to that shown in Muskhelishvili (1954). However, to develop a unified computer program valid for any kind of anisotropic materials, the degenerate materials are treated by introducing a small perturbation in the material properties. As to the real form solution shown in (6.4), no special numerical treatments or correspondence relations are needed for the degenerate materials since the solutions do not contain any material eigenvalues p_α or eigenvector matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$ explicitly. The reduction to the orthotropic and isotropic materials

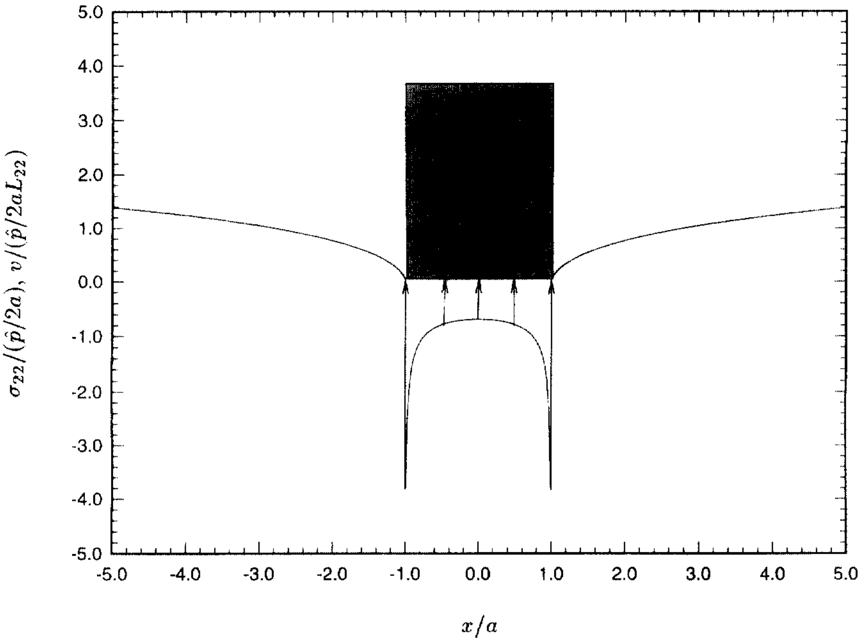

Fig. 1. Contact pressure and surface deformation for a flat-ended punch indenting into an orthotropic half-plane. (The oscillatory singularity zone near the ends of the punch is too small (about $10^{-4}a$) to be shown.)

for the contact pressure have been shown in (Fan and Hwu, 1996), whose results also agree with those shown in Muskhelisvili (1954).

Numerical example

Consider an orthotropic half-plane whose material properties are $E_1 = 60.7$ Gpa, $E_2 = 24.8$ Gpa, $G_{12} = 12.0$ Gpa, $v_{12} = 0.23$, where E, G and v are, respectively, the Young's modulus, shear modulus and the Poisson's ratio. The subscripts 1 and 2 denote the x and y directions. The contact region $2a$ is set to be 2 m of which the size is just a reference for the infinite domain. The contact pressure and the surface deformation are shown in Fig. 1. The stress singularity near the corners of the punch shown by eqn (6.4) can be found in this figure. However, since the oscillatory zone is too small (about 10^{-4} a), the oscillatory behavior near the corners of the punch cannot be revealed by this figure. To see the stresses in depth and to see the anisotropic effect, the contour plot of the nondimensionalized stress $\sigma_{22}/(\hat{p}/2a)$ shown in Fig. 2 may be helpful, which shows that the maximum stress along $x =$ constant occurs at a certain point under the surface not the point on the surface.

Note that when we plot the deformation outside the punch by using eqn $(6.4\text{a})_2$, an integration is necessary for finding $\underset{\sim}{u}$ from $\underset{\sim}{u}'$. However, the integration constant denoting the rigid body translation cannot be determined due to the infinite feature of our problem. To remedy this, we select the origin as a reference point whose displacement is set to be zero. By this way, the outlook of the surface deformation may be preserved without affecting our understanding of the physical behavior.

(ii) *Indentation tilted by a couple*

The second problem is a flat-ended punch ($n = 1$) tilted by the application of a couple m. The punch is of width $2a$ and is tilted through a small angle $\hat{\varepsilon}$ measured in the counterclockwise direction ($\underset{\sim}{\hat{u}}' = (0, \hat{\varepsilon}, 0)^T = \hat{\varepsilon}\underset{\sim}{i}_2$). The complex form solutions for the contact pressure and the relation between the applied couple m and the tilted angle $\hat{\varepsilon}$ have been given in (Fan and Hwu, 1996). Using a similar approach as the above subsection, we now list the explicit full field solution $\underset{\sim}{f}'(z)$, the real form solution for the contact pressure $\underset{\sim}{t}(x)$,

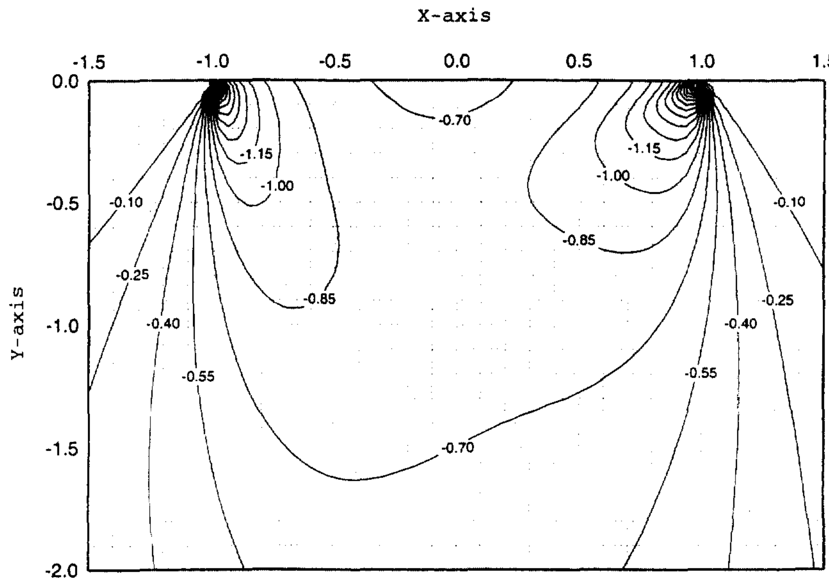

Fig. 2. Stress contour diagram of $\sigma_{22}/(\hat{p}/2a)$ for a flat-ended punch indenting into an orthotropic half-plane whose principal material axes are oriented 45° from the x–y coordinate axes.

and the real form solution for the surface deformation gradients $\mathbf{u}'(x)$ outside the punch. They are

$$\mathbf{f}'(z) = \frac{i\hat{\varepsilon}}{2}\left\{\mathbf{B}^{-1} - \sum_{k=1}^{3} \langle\langle \Gamma_k(z_\alpha)(z_\alpha + 2ia\varepsilon_k)\rangle\rangle \mathbf{B}^{-1}\mathbf{\Lambda}\mathbf{I}_k\mathbf{\Lambda}^{-1}\right\}\mathbf{L}\mathbf{i}_2,$$

$$\mathbf{t}(x) = \frac{\hat{\varepsilon}x}{\sqrt{a^2 - x^2}}\left[I + \frac{1 - c_R}{\beta^2}\mathbf{S}^{T^2} + \frac{c_I}{\beta}\mathbf{S}^T\right]\mathbf{L}\mathbf{i}_2, \quad |x| \leqslant a,$$

$$\mathbf{u}'(x) = \hat{\varepsilon}\left\{\mathbf{I} \mp \frac{x}{\sqrt{x^2 - a^2}}\left[I + \frac{1 - c_R^*}{\beta^2}\mathbf{S}^2 - \frac{c_I^*}{\beta}\mathbf{S}\right]\mathbf{i}_2\right\}, \quad x > a \quad \text{and} \quad x < -a, \tag{6.5a}$$

where

$$c_R + ic_I = \cosh(\pi\varepsilon)e^{-i\varepsilon\ln\frac{a+x}{a-x}}\left(1 + \frac{2ia\varepsilon}{x}\right),$$

$$c_R^* + ic_I^* = e^{-i\varepsilon\ln\left|\frac{x+a}{x-a}\right|}\left(1 + \frac{2ia\varepsilon}{x}\right). \tag{6.5b}$$

The real form relation between the applied couple m and the tilted angle α is found to be

$$m = \frac{\pi}{2}a^2\hat{\varepsilon}\left\{\left[I - \frac{4\varepsilon^2}{\beta^2}\mathbf{S}^{T^2}\right]\mathbf{L}\right\}_{22}. \tag{6.6}$$

Numerical example

Based upon the above formulae, the numerical calculation can also be done. Using the same material as subsection (i), setting the applied couple $m = 2 \times 10^6$ N·m, and calculating by (6.5), the contact pressure and the surface deformation of the present problem are

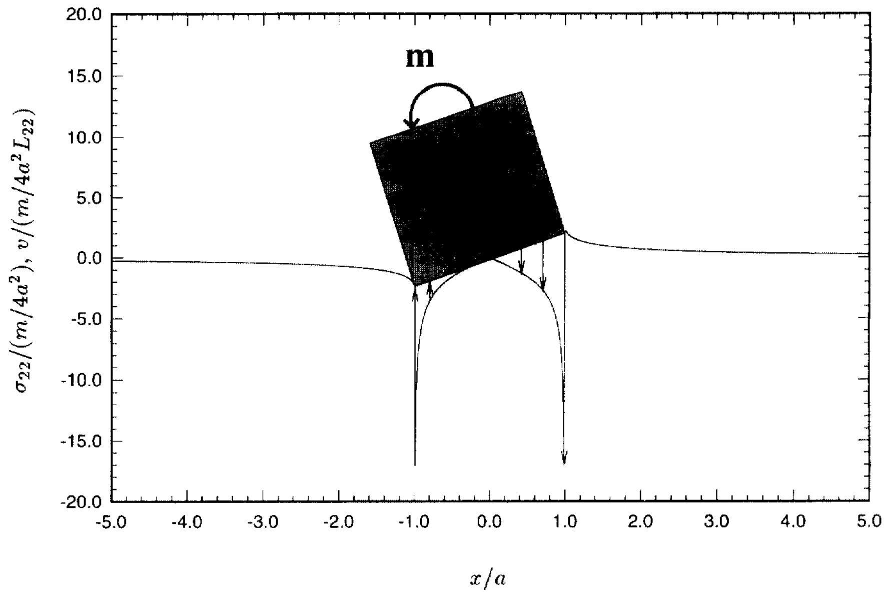

Fig. 3. Contact pressure and surface deformation for a flat-ended punch tilted by a couple. (The oscillatory singularity zone near the ends of the punch is too small (about $10^{-4}a$) to be shown.)

plotted in Fig. 3. From this figure, we see that the applied couple m induces tension on the right hand side and compression on the left hand side. It seems that the presence of negative pressure is wrong. However, the assumption that the punches completely adhere to the half-plane on the initial and subsequent indentation may allow this unrealistic result. Moreover, the oscillatory singularity behavior also occurs near the corners of the punches of which the region is too small to be shown in the figure. One more thing we would like to explain is that the negative and positive pressures under the punch seem to be unbalanced. By actually numerical integration along the punch region, we proved that it is really balanced, and this seemingly unbalance is due to the magnification of the rotation angle.

(iii) *Indentation by a parabolic punch*

Consider a symmetric punch whose end section can be expressed by a parabolic curve

$$y = \frac{x^2}{2R}, \quad |x| \leqslant l, \tag{6.7}$$

where R is the radius of curvature, and $2l$ is the width of the punch. Let us suppose on indentation under the force $\hat{p}$ the size of the contact region is $2a(\leqslant 2l)$. By the assumption that the punch completely adheres to the half-plane, we have

$$\hat{u}'(x) = \frac{x}{R} i_2, \quad |x| \leqslant a. \tag{6.8}$$

Substituting (6.8) and (4.15) into $(4.12a)_2$, we find

$$\theta'(z) = \frac{1}{2\pi R} X_0(z) \int_{-a}^{a} \frac{t}{t-z} [X_0^+(t)]^{-1} \, \mathrm{d}t \bar{M} i_2 + \frac{1}{2\pi i} X_0(z) \Lambda^{-1} \hat{p}. \tag{6.9}$$

The line integral in (6.9) can be evaluated by a way similar to that presented in (Fan and Hwu, 1996). The result is

$$\int_{-a}^{a} \frac{t}{t-z}[\underset{\sim}{X}_0^+(t)]^{-1}\,\mathrm{d}t\underset{\sim}{\bar{M}}\underset{\sim}{i}_2 = \pi i\left\{z[\underset{\sim}{X}_0(z)]^{-1} - \left\langle\left\langle z^2+2ia\varepsilon_\alpha z - \left(\frac{1}{2}+2\varepsilon_\alpha^2\right)a^2\right\rangle\right\rangle \underset{\sim}{\Lambda}^{-1}\right\}\underset{\sim}{L}\underset{\sim}{i}_2. \tag{6.10}$$

Substituting (6.10) and (4.13) into (6.9), we have

$$\underset{\sim}{\theta}'(z) = \frac{i}{2R}\left\{z\underset{\sim}{I}-\underset{\sim}{\Lambda}\underset{\sim}{\Gamma}(z)\left\langle\left\langle z^2+2ia\varepsilon_\alpha z-\left(\frac{1}{2}+2\varepsilon_\alpha^2\right)a^2\right\rangle\right\rangle\underset{\sim}{\Lambda}^{-1}\right\}\underset{\sim}{L}\underset{\sim}{i}_2+\frac{1}{2\pi i}\underset{\sim}{\Lambda}\underset{\sim}{\Gamma}(z)\underset{\sim}{\Lambda}^{-1}\underset{\sim}{\hat{p}}, \tag{6.11}$$

where $\underset{\sim}{\Gamma}(z)$ is the same as (6.1b). By (6.11), $(4.12\text{a})_1$ and (2.4), the explicit full field solution can be expressed as

$$\underset{\sim}{f}'(z) = \frac{i}{2R}\langle\langle z_\alpha\rangle\rangle\underset{\sim}{B}^{-1}\underset{\sim}{L}\underset{\sim}{i}_2 - \frac{i}{2R}\sum_{k=1}^{3}\left\langle\left\langle \Gamma_k(z_\alpha)\left[z_\alpha^2+2ia\varepsilon_k z_\alpha-\left(\frac{1}{2}+2\varepsilon_k^2\right)a^2\right]\right\rangle\right\rangle\times$$

$$\underset{\sim}{B}^{-1}\underset{\sim}{\Lambda}\underset{\sim}{I}_k\underset{\sim}{\Lambda}^{-1}\underset{\sim}{L}\underset{\sim}{i}_2+\frac{1}{2\pi i}\sum_{k=1}^{3}\langle\langle\Gamma_k(z_\alpha)\rangle\rangle\underset{\sim}{B}^{-1}\underset{\sim}{\Lambda}\underset{\sim}{I}_k\underset{\sim}{\Lambda}^{-1}\underset{\sim}{\hat{p}}. \tag{6.12}$$

The real form solutions for the contact pressure and the surface deformation may also be obtained by the way similar to that presented in subsection (i). The results are

$$\underset{\sim}{t}(x) = \frac{2x^2-a^2}{2R\sqrt{a^2-x^2}}\left[\underset{\sim}{I}+\frac{1-(c\tilde{c})_R}{\beta^2}\underset{\sim}{S}^{T^2}+\frac{(c\tilde{c})_I}{\beta}\underset{\sim}{S}^T\right]\underset{\sim}{L}\underset{\sim}{i}_2$$

$$+\frac{1}{\pi\sqrt{a^2-x^2}}\left[\underset{\sim}{I}+\frac{1-c_R}{\beta^2}\underset{\sim}{S}^{T^2}+\frac{c_I}{\beta}\underset{\sim}{S}^T\right]\underset{\sim}{\hat{p}},\quad |x|\leqslant a, \tag{6.13a}$$

$$\underset{\sim}{u}'(x) = \frac{x}{R}\underset{\sim}{i}_2\mp\frac{2x^2-a^2}{2R\sqrt{x^2-a^2}}\left[\underset{\sim}{I}+\frac{1-(c^*\tilde{c})_R}{\beta^2}\underset{\sim}{S}^2-\frac{(c^*\tilde{c})_I}{\beta}\underset{\sim}{S}\right]\underset{\sim}{i}_2$$

$$\mp\frac{1}{\pi\sqrt{x^2-a^2}}\underset{\sim}{L}^{-1}\left[\underset{\sim}{I}+\frac{1-c_R^*}{\beta^2}\underset{\sim}{S}^{T^2}+\frac{c_I^*}{\beta}\underset{\sim}{S}^T\right]\underset{\sim}{\hat{p}},\quad x>a\ \text{and}\ x<-a, \tag{6.13b}$$

where

$$c_R+ic_I = \cosh(\pi\varepsilon)e^{-i\varepsilon\ln\frac{a+x}{a-x}},$$

$$c_R^*+ic_I^* = e^{-i\varepsilon\ln\left|\frac{x+a}{x-a}\right|},$$

$$\tilde{c}_R+i\tilde{c}_R = \frac{1}{2x^2-a^2}[2x^2-(1+4\varepsilon^2)a^2+i4a\varepsilon x], \tag{6.13c}$$

and

$$(c\tilde{c})_R = c_R\tilde{c}_R-c_I\tilde{c}_I,\quad (c\tilde{c})_I = c_R\tilde{c}_I+c_I\tilde{c}_R,$$

$$(c^*\tilde{c})_R = c_R^*\tilde{c}_R-c_I^*\tilde{c}_I,\quad (c^*\tilde{c})_I = c_R^*\tilde{c}_I+c_I^*\tilde{c}_R. \tag{6.13d}$$

Note that during the derivation of (6.13b), the identities $\underset{\sim}{L}^{-1}\underset{\sim}{S}^T\underset{\sim}{L} = -\underset{\sim}{S}$ and $\underset{\sim}{L}^{-1}\underset{\sim}{S}^{T^2}\underset{\sim}{L} = \underset{\sim}{S}^2$ (Ting, 1986) have been used.

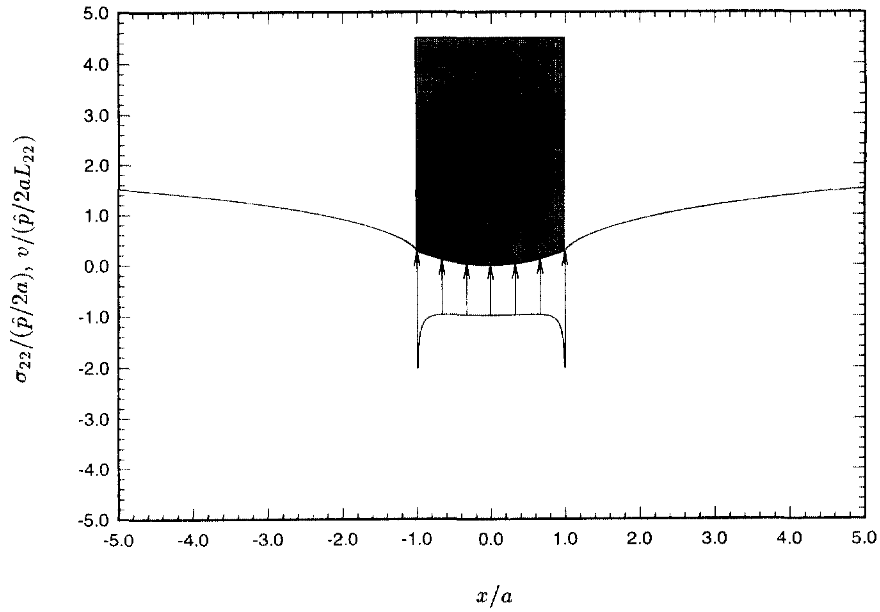

Fig. 4. Contact pressure and surface deformation for a parabolic punch indenting into an orthotropic half-plane. (The oscillatory singularity zone near the ends of the punch is too small (about $10^{-4}a$) to be shown.)

Numerical example

The material properties for the half-plane used in this example are the same as those given in subsection (i). The force $\hat{p}$ applied on the punch is in the vertical direction and equals to $5.0 \times 10^8 Nt$. The size of the contact region $2a$ is equal to the punch width $2l$ where $l = 1$ m, and the radius of curvature R of the parabolic punch is $R = 100$ m of which the size is just a reference for the infinite domain. By using the real form solutions shown in (6.13), the results of the contact pressure and surface deformation are shown in Fig. 4. As expected, the stress singularity also occurs near the corners of the punch.

7. CONCLUSIONS

In this paper, the punch problems are solved by using the solutions of the collinear interface crack problems with one of the materials taken to be rigid and the material eigenvector matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ interchanged. The solutions found by this analogy have been proved to be identical to those given in the literature. Besides this analogy, very simple expressions for the contact pressure and the surface deformation have been derived. By some more analogies, the explicit full field solutions, the real form solutions for the contact pressure and the surface deformation are derived for three representative punch problems. Based upon these closed-form solutions, several numerical examples and illustrating figures were done. It is also hoped that the analogy concept may be extended to solve the problems of elastic contact between two dissimilar anisotropic media.

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REFERENCES

Brock, L. M. (1978) Dynamics analysis of non-symmetric problems for frictionless indentation and plane crack extension. *Journal of Elasticity* **8**(3), 273–283.

Eshelby, J. D., Read, W. T. and Shockley, W. (1953) Anisotropic elasticity with applications to dislocation theory. *Acta Metallica* **1**, 251–259.

Fan, C. W. and Hwu, C. (1996) Punch problems for an anisotropic elastic half-plane. *Journal of Applied Mechanics* **63**(1), 69–76.

Gladwell, G. M. L. (1980) *Contact Problems in the Classical Theory of Elasticity*. Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands.

Hwu, C. (1993a) Explicit solutions for the collinear interface crack problems. *International Journal of Solids and Structures* **30,** 301–312.

Hwu, C. (1993b) Fracture parameters for orthotropic bimaterial interface cracks. *Engineering Fracture Mechanics* **45**(1), 89–97.

Hwu, C. (1996) Correspondence relations between anisotropic and isotropic elasticity. *Chinese Journal of Mechanics* **12**(4), 483–493.

Hwu, C. and Yen, W. J. (1993) On the anisotropic elastic inclusions in plane elastostatics. *ASME Journal of Applied Mechanics* **60,** 626–632.

Lekhnitskii, S. G. (1963) *Theory of Elasticity of an Anisotropic Body*. MIR, Moscow.

Muskhelishvili, N. I. (1954) *Some Basic Problems of the Mathematical Theory of Elasticity*. Noordhoff Pub., Groningen.

Sneddon, I. N. (1946) Boussinesq's problem for a flat-ended cylinder. *Proceedings of Cambridge Philosophical Society* **42,** 29–39.

Sneddon, I. N. and Lowengrub, M. (1969) *Crack Problems in the Classical Theory of Elasticity*. John Wiley, New York.

Stroh, A. N. (1958) Dislocations and cracks in anisotropic elasticity. *Philosophical Magazine* **7,** 625–646.

Suo, Z. (1990) Singularities, interfaces and cracks in dissimilar anisotropic media. *Proceedings of the Royal Society London Series A* **427,** 331–358.

Timoshenko, S. P. and Goodier, J. N. (1970) *Theory of Elasticity*. McGraw-Hill Book Company, New York.

Ting, T. C. T. (1986). Explicit solution and invariance of the singularities at an interface crack in anisotropic composites. *International Journal of Solids and Structures* **22,** 965–983.

Ting, T. C. T. (1988) Some identities and the structures of $\tilde{N}_i$ in the Stroh formalism of anisotropic elasticity. *Quarterly Journal of Applied Mathematics* **46,** 109–120.

Ting, T. C. T. (1996) *Anisotropic Elasticity—Theory and Applications*. Oxford Science Publications, New York.

Willis, J. R. (1966) Hertzian contact of anisotropic bodies. *Journal of the Mechanics and Physics of Solids* **14,** 163–176.

Willis, J. R. (1967) Boussinesq problems for an anisotropic half-space. *Journal of the Mechanics and Physics of Solids* **15,** 331–339.

Willis, J. R. (1968) The stress field around an elliptical crack in an anisotropic elastic medium. *International Journal of Engineering Science* **6,** 253–263.

Willis, J. R. (1970) The distribution of stress in an anisotropic elastic body containing an exterior crack. *International Journal of Engineering Science* **8,** 559–574.